

# THE QUANTUM $\mathfrak{sl}(3)$ INVARIANTS OF CUBIC BIPARTITE PLANAR GRAPHS

DONGSEOK KIM AND JAEUN LEE

**ABSTRACT.** Temperley-Lieb algebras have been generalized to  $\mathfrak{sl}(3, \mathbb{C})$  web spaces. Since a cubic bipartite planar graph with suitable directions on edges is a web, the quantum  $\mathfrak{sl}(3)$  invariants naturally extend to all cubic bipartite planar graphs. First we completely classify them as a connected sum of primes webs. We also provide a method to find all prime webs and exhibit all prime webs up to 20 vertices. Using quantum  $\mathfrak{sl}(3)$  invariants, we provide a criterion which determine the symmetry of graphs.

## 1. INTRODUCTION

Triangulations are maximal simple planar graphs where we can not add an edge without destroying the planarity. Every triangulations are 3-connected, thus it has a unique embedding into the two dimensional sphere [31]. The dual of a triangulation is a 3-connected cubic planar graphs. If a cubic planar graph is not 3-connected, it could have several nonequivalent embeddings. Decomposing these cubic planar graphs by the connectivity has powerful applications such as to count all cubic planar graphs [30]. The Barnette's conjecture, every 3-connected cubic bipartite planar graphs are Hamiltonian [2], brought a lot of attentions to cubic bipartite planar graphs [8].

An unexpected relation between the representation theory and bipartite cubic planar graphs has disclosed as follows. After the discovery of the Jones polynomial [11], its generalizations have been studied in many different ways. One of successful generalizations is to use the representation theory of complex simple Lie algebras from the original work of Reshetikhin and Turaev [27]. In particular, Kuperberg generalized Temperley-Lieb algebras, which corresponds to the invariant subspace of a tensor product of the vector representation of  $\mathfrak{sl}(2)$ , to web spaces of simple Lie algebras of rank 2,  $\mathfrak{sl}(3)$ ,  $\mathfrak{sp}(4)$  and  $G_2$  [17]. All webs in the representation theory of  $\mathfrak{sl}(3)$  are generated by the webs in Figure 1 with a complete set of the relations presented in Figure 2 where the quantum integers are defined as

$$[n] = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}.$$

Roughly speaking, a *basis web* is a web for which we can not apply any relation in Figure 2. For precise definition, we refer to [17]. There are many interesting results on the quantum  $\mathfrak{sl}(3)$  invariants [6, 14–16, 20, 21, 24, 25, 28].

The  $\mathfrak{sl}(3)$  webs are directed cubic bipartite planar graphs together with circles (without vertices, thus different from loops) where the direction of the edges is from one set to the other set in the bipartition. From a given cubic bipartite planar graph, one can find its quantum  $\mathfrak{sl}(3)$  invariant as follows. Once we direct the edges by one of two possible orientations

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FIGURE 1. Generators of the quantum  $\mathfrak{sl}(3)$  web space.

$$\begin{aligned}
 (1) \quad & \text{A circle with a clockwise arrow} = [3] \\
 (2) \quad & \text{A horizontal line with a loop (two arrows forming a circle) in the middle} = -[2] \text{ followed by a horizontal line with an arrow pointing left} \\
 (3) \quad & \text{A square with arrows on all four sides (top: left to right, bottom: right to left, left: bottom to top, right: top to bottom)} = \text{Two curved arrows (one top, one bottom)} + \text{Two curved arrows (one left, one right)}
 \end{aligned}$$

FIGURE 2. Complete relations of the quantum  $\mathfrak{sl}(3)$  web space.

as we described, it can be considered a web of empty boundary. Since the dimension of the web space of webs of empty boundary is one over  $\mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ , we obtain a Laurent polynomial. In Lemma 2.1, we prove the quantum  $\mathfrak{sl}(3)$  invariant does not depend on the choice of directions. Therefore, the quantum  $\mathfrak{sl}(3)$  invariant naturally extends to any cubic bipartite planar graph  $G$ , let us denote it by  $P_G(q)$ . If a cubic bipartite planar graph  $G$  is a disjoint union of two cubic bipartite planar graphs  $G_1$  and  $G_2$ , then  $P_G = P_{G_1} P_{G_2}$ . Thus, we assume all abstract cubic bipartite planar graphs are connected. Then we find the following classification theorem.

**Theorem 1.1.** *Let  $G$  be a connected cubic bipartite planar graph. Then there exist 3-connected cubic bipartite planar graphs  $G_1, G_2, \dots, G_k$  such that*

$$G = G_1 \# G_2 \# \dots \# G_k,$$

where the  $\#$  operation is defined in Figure 3. Moreover, the decomposition does not depend the choice of the planar imbedding of  $G$  up to a reflection of  $G_i$ 's on the two dimensional sphere, thus the decomposition is unique up to the reflections of  $G_i$ . For the quantum  $\mathfrak{sl}(3)$  graph invariant of  $G$ , we find

$$[3]^{k-1} P_G(q) = (-[2])^l P_{G_1}(q) P_{G_2}(q) \dots P_{G_k}(q),$$

where  $l$  is the number of times we use relation 2 shown in Figure 2 in the process of the decomposition.

The outline of this paper is as follows. In section 2 we first classify cubic bipartite planar graphs and show  $P_G$  can be computed by the decomposition. Using the representation theory, we provide a method to find all prime webs and exhibit all prime webs up to 20 vertices in section 3. We provide a criterion which determine the symmetry of cubic bipartite planar graphs and we discuss a few problems in section 4.

## 2. A CLASSIFICATION OF CUBIC BIPARTITE PLANAR GRAPHS

First we prove the quantum  $\mathfrak{sl}(3)$  invariants can be defined for (undirected) cubic bipartite planar graphs. For terms and notations for graph theory, we refer to [9].

**Lemma 2.1.**  *$P_G(q)$  does not depend on the direction of the edges.*

*Proof.* From a given connected cubic bipartite planar graph, there are exactly two possible ways to direct the edges to get a web and one can be obtained from the other by reversing the directions of all edges. The first two relations 1, 2 shown in Figure 2 can be applied exactly same way if we reverse the orientations of all edges. For the relation 3, it does not change how the rectangle splits but it only changes the direction of the edges. Therefore,  $P_G$  does not depend on the direction of the edges.  $\square$

By comparing these three relations for the mirror image, we find the following lemma.

**Lemma 2.2.** *Let  $G$  be a cubic bipartite planar graph and let  $\tilde{G}$  be the mirror image of  $G$ . Then,*

$$P_G(q) = P_{\tilde{G}}(q).$$

Let  $G$  be a web of empty boundary. As we mentioned,  $G$  is a cubic bipartite directed planar graph where the directions of edges are from one set to the other set in the definition of the bipartition. By Lemma 2.1 we assume one of these directions on the edges when we say a cubic bipartite planar graph. Because of the directions of edges, there does not exist a loop (it is different from the circle without a vertex). Since we can remove all multiple edges by relation 2 presented in Figure 2, we assume  $G$  is simple. The *connectivity*  $\kappa(G)$  of a graph  $G$  is the minimum number of vertices whose removal results in a disconnected graph or a single vertex. A graph is *n-connected* if the connectivity of  $G$  is  $n$  or greater. A graph  $G$  is *k-edge-connected* if the removal of fewer than  $k$  edges from  $G$  still leaves a connected graph, we denote the edge connectivity of  $G$  by  $\kappa'(G)$ . For an  $r$ -regular graph  $G$ ,  $\kappa(G) \leq \kappa'(G) \leq r$ , hence if  $\kappa(G) = r$ , then  $\kappa(G) = \kappa'(G) = r$ . For connected trivalent graphs, these two connectivities are the same with one exception  $\Theta$ , the base graph of the covering in Figure 4. Since it is not simple and we are going to deal with only simple graphs, now we can assume  $\kappa(G) = \kappa'(G)$ . Since it is cubic, its connectivity is less than or equal to 3. If it is 3-connected, by a celebrated theorem by Whitney, there is only one way to imbed  $G$  into the plane, up to a reflection [31]. A cubic bipartite planar graphs is *prime* if it is 3-connected. Conventionally, we assume the circle without a vertex is not prime. If  $G$  is not 3-connected, then we want to decompose  $G$  into a smaller pieces of prime graphs.

**Lemma 2.3.** *Let  $G$  be a connected cubic bipartite planar graph. Then  $G$  is either 3-connected or 2-connected. Moreover, if it is 2-connected, then two edges, whose removal results two disconnect graphs, intersect the separating circle, presented by a thick gray line, in alternating directions as in Figure 3.*

*Proof.* One can prove it by using a method in graph theory. But if we use the representation theory, one can easily prove the result. In the language of representation theory, the lemma can be restated that there does not exist a cut circle of the weights  $\lambda_1, \lambda_2, 2\lambda_1$  or  $2\lambda_2$ . Since  $\dim(\text{Inv}(V_{\lambda_i})) \neq 0$ ,  $G$  is not 1-connected. Since  $(V_{\lambda_1})^* \cong V_{\lambda_2}$ ,  $(V_{\lambda_2})^* \cong V_{\lambda_1}$ , by a simple application of Schur's lemma, we find

$$\dim(\text{Inv}(V_{\lambda_i} \otimes (V_{\lambda_j})^*)) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

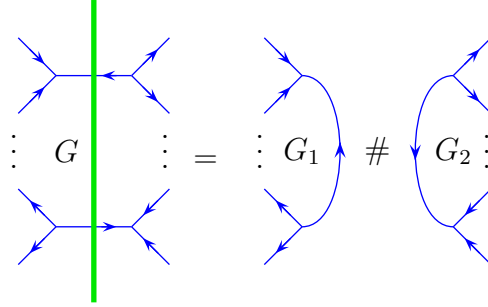


FIGURE 3. A connected sum decomposition of a 2-connected cubic bipartite planar graph  $G$  into  $G_1 \# G_2$ .

Moreover, if  $G$  is 2-connected, then the weight of the cut circle must be  $\lambda_1 + \lambda_2$ .  $\square$

**2.1. Proof of Theorem 1.1.** If  $G$  is 2-connected, then its general shape is given in the left hand side of the equality in Figure 3. Then we naturally define a connected sum of two graphs (depend on the choice of the edges). The resulting graphs  $G_1$  and  $G_2$  are obviously cubic bipartite planar graphs but they may not be simple. We apply relation 2 shown in Figure 2 and resulting one still may not be simple but we repeat the relation until it becomes simple. From these two components  $G_1$  and  $G_2$ , we repeat the process. 3-connected parts are not changed by relation 2 shown in Figure 2, thus there is no ambiguity for the order of connected sums and the number of times we use the relation 2 does not depend neither. The finiteness of  $G$  implies that the process stops and we obtain a unique prime decomposition of  $G$ . The following equality immediately follows from the decomposition process,

$$[3]^{k-1} P_G(q) = (-[2])^l P_{G_1}(q) P_{G_2}(q) \dots P_{G_k}(q).$$

One can see that this connected sum does depend on the choice of the edges we connect two graphs. Thus,  $G_1 \# G_2$  is not unique in general. Using this idea, one can construct non-isomorphic non-prime graphs of the same quantum  $\mathfrak{sl}(3)$  invariant.

### 3. PRIME CUBIC BIPARTITE PLANAR GRAPHS

For a fixed boundary, there is a systematic way to generate all webs [18]. But we are only interested in prime webs, thus we develop a new way to produce all prime webs in the section. As a cubic bipartite graph, all prime webs are graph coverings of the graph  $\Theta$  [19]. Since the dual graph of a web is a planar triangulations that the valences of all vertices are even, this dual graph is vertex 3-colorable, *i.e.*, cubic bipartite planar graphs are edge 3-colorable. In fact, the number of such colorings of an (undirected) cubic bipartite planar graph is known [10]. Thus, each polygons of a web can be alternatively colored by just two colors naturally come from the 3-coloring of the dual graph. From such a coloring, we can consider a web as a union of polygons colored by two colors and these polygons are connected by the edges of a color which has not been used yet, we called it a *polygonal decomposition* of the web. Since there are three colors, we find exactly three polygonal decompositions of a web as in Figure 5. In fact, the first two are identical up to a permutation of colors.

In particular, some polygonal decompositions can be obtained from the representation theory of the quantum  $\mathfrak{sl}(2)$ . First, we fix a face which is not a polygon in a polygonal

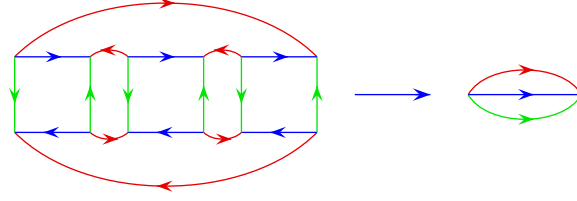


FIGURE 4. A graph covering structure of a prime web.

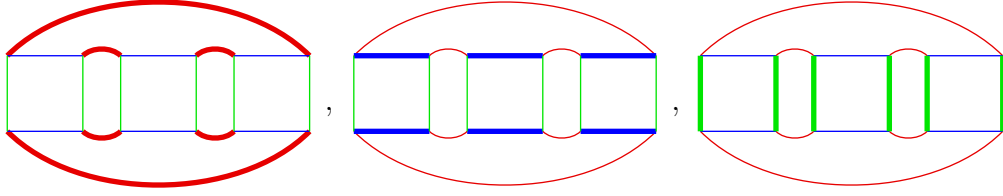


FIGURE 5. Three polygonal decompositions of the prime web in Figure 4.

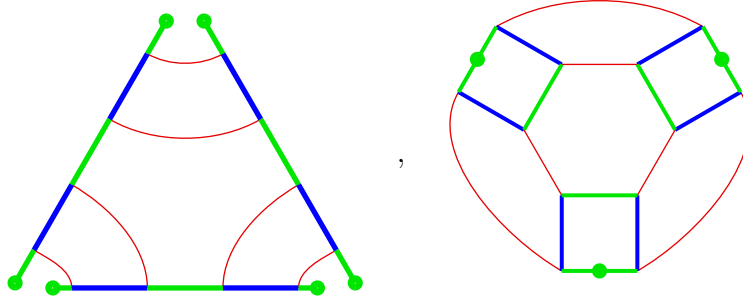


FIGURE 6. A normal chord diagram and its corresponding polygonal decomposition in Figure 5.

decomposition of a web, we call it an *exterior* face. For each polygon in a polygonal decomposition, we define the *level* of a polygon by the minimum of the faces, it has to cross to reach the exterior face. In fact, the level of a polygon is the minimal *length* of paths between vertices in the dual graph, where vertices are corresponding to the polygon and the exterior face. A polygonal decomposition of a web is *circular* if there exists a suitable exterior face such that all polygons in the decomposition has level 1. A web is *circular* if at least one of polygonal decompositions of the web is circular. All three in Figure 5 are circular, so is the web.

All circular prime webs of a fixed  $N$  tuple of even integers  $(a_1, a_2, \dots, a_N)$ , where  $a_i \geq 4$ , can be found the following way. Because of 3-connectivity of prime webs, for each polygon in a circular polygonal decomposition, there exists a unique edge between the polygon and the exterior face. Then we cut open each polygon along this unique edge. Then we have a polygon of size  $N$  with  $a_i$  points marking on sides in cyclic order, let us call it a *plate* and  $N$  is the *size* of the plate. The edges of the color which has not been used in polygonal decomposition give us a chord diagram of the plate. Moreover, this chord diagram is called *normal* if there is no chord connecting points in the same edge of the plate. Normal chord diagrams are well understood by the representation theory of  $\mathfrak{sl}(2)$ . We will explain normal

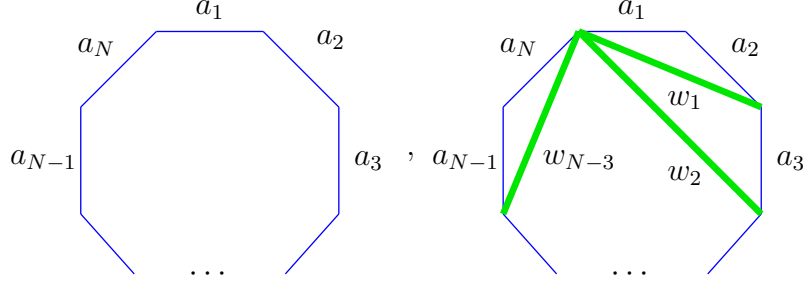


FIGURE 7. A normal chord diagram interpolation of equation 4.

chord diagrams by the size of the plate. If  $N = 2$ , the only possible normal chord diagram of the plate of size 2 exists if  $a_1$  is equal to  $a_2$ . Thus if we fixed the number of the vertices of the web, the prime web which has a polygonal decomposition of two polygons is unique. For  $N = 3$ , we say  $(a_1, a_2, a_3)$  is an *admissible triple* if  $|a_1 - a_2| \leq a_3 \leq a_1 + a_2$ . Let us remind that all  $a_i$  are even integers. For an admissible triple, there exists a unique normal chord diagram. In the language of representation theory [13, 29], let  $V_{a_1}$  be an irreducible representation of  $\mathfrak{sl}(2)$  of highest weight  $a_1$ , then

$$\dim(\text{Inv}(V_{a_1} \otimes V_{a_2} \otimes V_{a_3})) = \begin{cases} 1 & \text{if } (a_1, a_2, a_3) \text{ is an admissible triple,} \\ 0 & \text{otherwise.} \end{cases}$$

Let  $\mathcal{C}$  be the set of all irreducible representations of  $\mathfrak{sl}(2)$ . For  $N \geq 4$ , we use the following fact,

$$\begin{aligned} \dim(\text{Inv}(V_{a_1} \otimes V_{a_2} \otimes \dots \otimes V_{a_N})) &= \sum_{V_{w_1} \in \mathcal{C}} \dots \sum_{V_{w_{N-3}} \in \mathcal{C}} \dim(\text{Inv}(V_{a_1} \otimes V_{a_2} \otimes V_{w_1})) \\ (4) \quad &\dim(\text{Inv}(V_{w_1} \otimes V_{a_3} \otimes V_{w_2})) \dots \dim(\text{Inv}(V_{w_{N-3}} \otimes V_{a_{N-1}} \otimes V_{a_N})). \end{aligned}$$

Geometrically, this can be interpolated that we can count all normal chord diagrams of fixed boundary as a sum of products of the normal chord diagrams of triangles where the sums run on  $\mathcal{C}$  by the number of chords passing the line which represents an irreducible representation  $V_{w_i}$  as in Figure 7. Even though  $\mathcal{C}$  has infinitely many elements, the actual sums run only on finitely many terms because there are only finitely many admissible triples once we fix two entries of the triples by the Clebsch-Gordan theorem [7].

For example, let us find all circular webs of 18 vertices. Since the sizes of every polygons are even and bigger than or equal to 4, we first find all even integral partitions of a given number of vertices which satisfy these assumptions. For 18, there are three possible plate sizes, 2, 3 and 4. But  $N = 2$  can not be happened because  $\frac{18}{2}$  is not even. For  $N = 3$ , we have three even integral partitions of 18 of length 3,  $(4, 4, 10)$ ,  $(4, 6, 8)$  and  $(6, 6, 6)$ . But we can exclude  $(4, 4, 10)$  because there does not exist a normal chord diagrams of a triangle with  $(4, 4, 10)$  markings on the edges, or one can check  $\dim(\text{Inv}(V_4 \otimes V_4 \otimes V_{10})) = 0$ . We find the chord diagrams of  $(4, 6, 8)$  and  $(6, 6, 6)$  in Figure 8. The chord diagram of  $(6, 6, 6)$  makes the prime web  $9_1$  and the chord diagram of  $(4, 6, 8)$  makes the prime web  $9_2$  in Figure 12. For  $N = 4$ , the only even integral partition of 18 of length 4 is  $(4, 4, 4, 6)$ . From equation 4, we find that there are four possible nonzero cases  $(4, 4, 4) \cdot (2, 4, 6)$ ,  $(4, 4, 4) \cdot (4, 4, 6)$ ,  $(4, 4, 6) \cdot (6, 4, 6)$  and  $(4, 4, 8) \cdot (8, 4, 6)$ . But one can easily see that the first and the last are

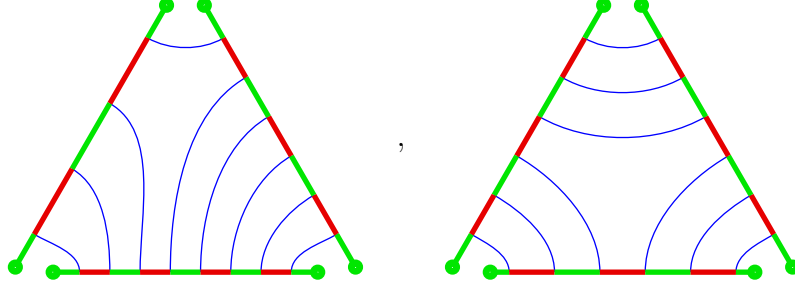
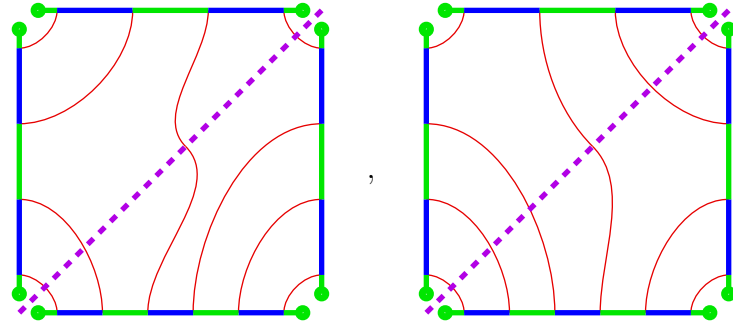
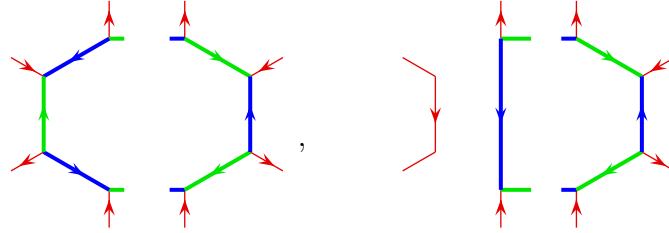
FIGURE 8. Normal chord diagrams of  $(4, 6, 8)$  and  $(6, 6, 6)$ .FIGURE 9. Normal chord diagrams of  $(4, 4, 4) \cdot (4, 4, 6)$  and  $(4, 4, 6) \cdot (6, 4, 6)$ .

FIGURE 10. A pushing move.

not prime. For the other we find the chord diagrams of  $(4, 4, 4) \cdot (4, 4, 6)$  and  $(4, 4, 6) \cdot (6, 4, 6)$  in Figure 9. Both chord diagrams make the prime web  $9_2$  in Figure 12.

To deal with non-circular webs, we define a *pushing move* as shown in Figure 10. Then we find all non-circular webs can be obtained by a finite sequence of pushing moves from circular webs. First one can see any non-circular polygonal decompositions of a web has a polygon of level 2. The local shape around a polygon of level 2 is given in the left hand side of Figure 11. One can see that there exists at least one polygon of the same color type adjacent to the face between the level 2 polygon and the exterior polygon. Then if we do a converse of a pushing move, then the sizes of new polygons,  $*$  in Figure 11, are bigger than or equal to 4. Thus the resulting one is again a prime web. To finish the proof, we induct on the number of polygons of level bigger than 1 in a non-circular polygonal decomposition of a non-circular prime web. For a polygon of level 2, we can see it can be obtained by a

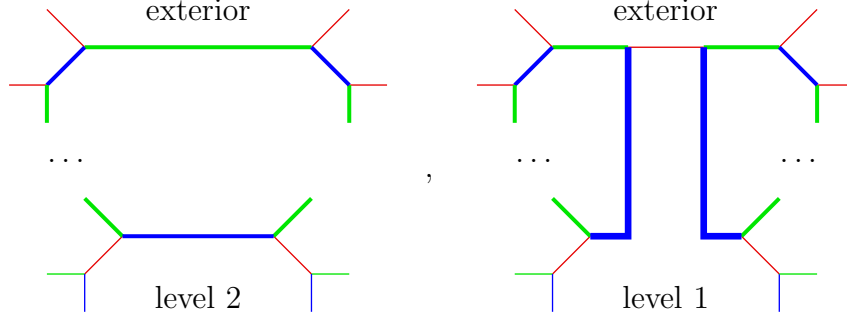


FIGURE 11. A local shape around a polygon of level 2 and a converse of a pushing move.

pushing move from a polygonal decomposition with one less polygons of level bigger than 1. Inductively, it completes the proof.

For a fixed number  $N$  of vertices, there is an upper bound  $f(N)$  such that all prime webs of  $N$  vertices can be found from circular prime webs of  $f(N)$  vertices. One of the easiest estimation is  $N + \lceil \frac{N}{8} \rceil$  because at each level we must have at least two polygons. In fact, this can be improved drastically because  $f(20)$  is just 22.

**3.1. Prime cubic bipartite planar graphs up to 20 vertices.** Using the method described the above, we can find a list of all prime webs up to 20 vertices in Figure 12. In Table 1 and 2, we list their quantum  $\mathfrak{sl}(3)$  invariants, three polygonal descriptions and circularness of prime webs up to 20 vertices. It is easy to find all circular webs up to 22 vertices, there are 8 circular prime webs of 22 vertices, surprisingly these are all prime webs of 22 vertices too. From these webs, we find all non-circular prime webs up to 20 vertices.

#### 4. APPLICATIONS AND DISCUSSIONS

**4.1. Symmetry of graphs and quantum  $\mathfrak{sl}(3)$  invariants.** A link  $L$  in  $S^3$  is  $n$ -periodic if there exists a periodic homeomorphism  $h$  of order  $n$  such that  $fix(h) \cong S^1$ ,  $h(L) = L$  and  $fix(h) \cap L = \emptyset$  where  $fix(h)$  is the set of fixed points of  $h$ . By a positive answer of the Smith conjecture, if we consider  $S^3$  as  $\mathbb{R}^3 \cup \{\infty\}$ , we can assume that  $h$  is a rotation by  $2\pi/n$  angle around the  $z$ -axis. If  $L$  is a periodic link, we denote its quotient link by  $L/h$ . If a link  $L$  admits an orientation preserving action  $\Gamma$  of order  $n$ , then there are relations between the classical link polynomials such as the Alexander polynomial and the Jones polynomial of link  $L$  and its quotient link  $L/\Gamma$  [22, 23]. For quantum  $\mathfrak{sl}(3)$  links invariants, it was shown in [6]

$$P_L(q) \cong (P_{L/\Gamma}(q))^n \quad \text{modulo } \mathcal{I}_n,$$

where  $\mathcal{I}_n$  is the ideal of  $\mathbb{Z}[q^{\pm \frac{1}{2}}]$  generated by  $n$  and  $[3]^n - [3]$ . It has been generalized for the quantum  $\mathfrak{sl}(n)$  link invariants [12]. In fact, the original invariants belongs to  $\mathbb{C}[q^{\pm \frac{1}{2}}]$  but later it was shown that it really is a polynomial in  $\mathbb{Z}[q^{\frac{1}{2}} + q^{-\frac{1}{2}}]$  [20].

For planar cubic bipartite graphs, some of symmetries can be orientation reversing. But the idea of the proof [6] still works in general with one exception. If the fundamental domain of the action is a basis web with the given boundary, then there does not exist any relation



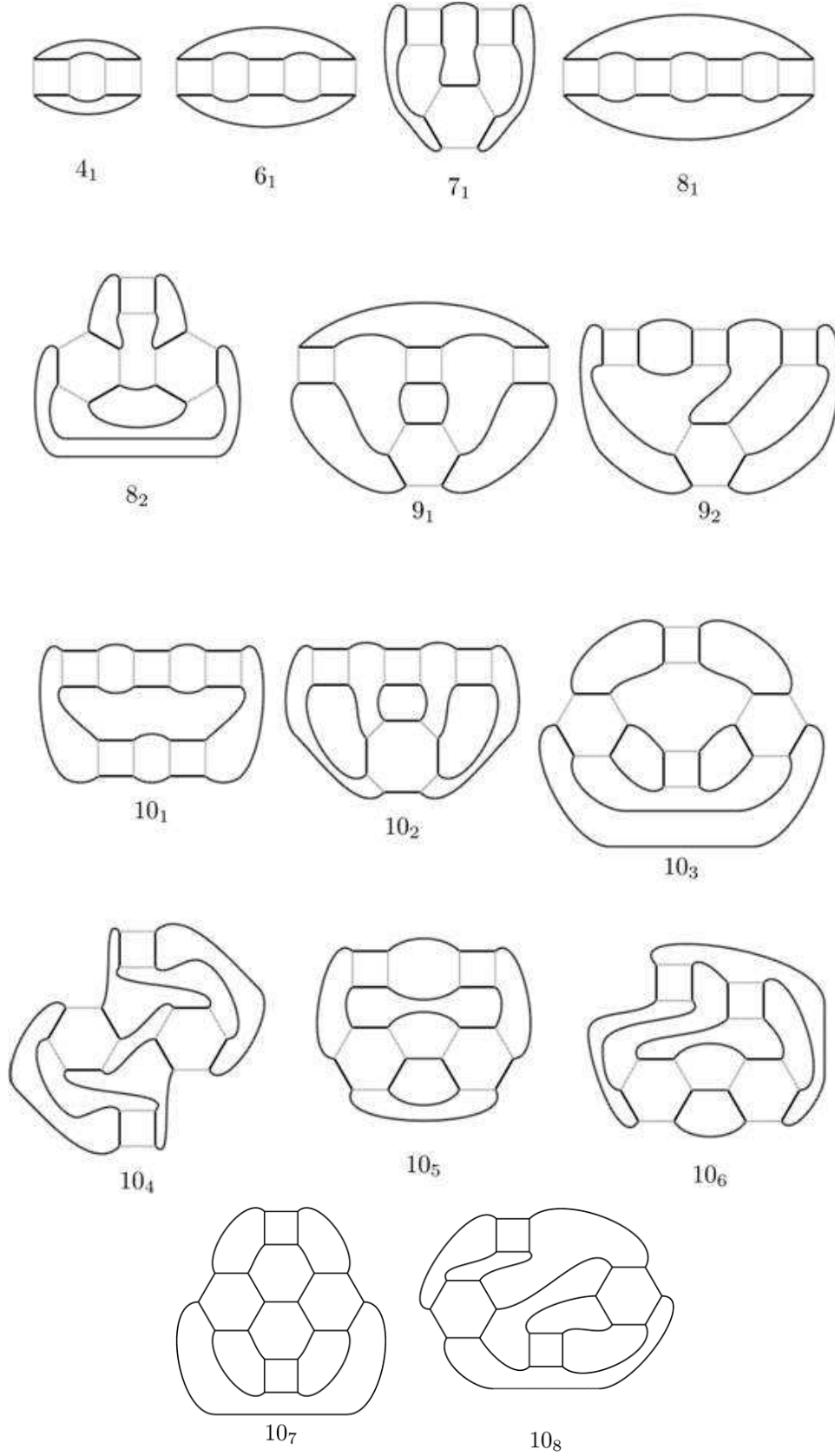


FIGURE 12. All prime cubic bipartite planar graphs up to 20 vertices.

or expansion which can be applied repeatedly, thus Theorem 4.1 might not work. In fact, we will demonstrate this in Example 4.2.

Let  $G$  be a planar cubic bipartite graph. Let  $\Gamma$  be the group of orientation preserving symmetries of  $G$ . Let  $G/\Gamma$  be the quotient graph of  $G$  by  $\Gamma$ . A direct relation between  $P_G(q)$  and  $P_{G/\Gamma}(q)$  is very difficult to find in general. Thus we will look at this relation modulo by an ideal in  $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ . Precisely we can state it as follows.

**Theorem 4.1.** *Let  $G$  be a planar cubic bipartite graph with the group of symmetries  $\Gamma$  of order  $n$ . Let  $\Gamma_d$  be a subgroup of  $\Gamma$  of order  $d$  such that the fundamental domain of  $G/\Gamma_d$  is not a basis web with the given boundary. Then*

$$P_G(q) \equiv (P_{G/\Gamma_d}(q))^d \quad \text{modulo } \mathcal{I}_d,$$

where  $\mathcal{I}_d$  is the ideal of  $\mathbb{Z}[q^{\pm\frac{1}{2}}]$  generated by  $d$  and  $[3]^d - [3]$ .

**Example 4.2.** *We look at an example  $6_1$ .*

There are twelve vertices but only six of them have the same directions. Thus, its all possible symmetries can have order either 6, 3 or 2. For  $n = 3$ , we find

$$\begin{aligned} [2]^4[3] + 2[2]^2[3] &\equiv [3]([3] + [1])^2 + 2([3] + [1]) \equiv [3]([3]^2 + 4[3] + 3[1]) \\ &\equiv [3]([3]^2 + [3]) \equiv [3]^3 + [3]^2 \equiv [3] + [3]^2 \\ &\equiv [3]([3] + [1]) \equiv [3]([3]^3 + 3[3]^2 + 3[3] + [1]) \\ &\equiv [3]([3] + [1])^3 \equiv [3]^3[2]^6 \equiv ([3][2]^2)^3 \pmod{\mathcal{I}_3} \end{aligned}$$

In fact, it does have a symmetry of order three by a rotation along a point in a hexagon. For  $n = 2$ , we find

$$[2]^4[3] + 2[2]^2[3] \equiv ([2]^2[3])^2 \pmod{\mathcal{I}_2}.$$

Also it does have a symmetry of order two by a rotation along a point in a rectangle. The quantum  $\mathfrak{sl}(3)$  invariant of the quotient of  $6_1$  is  $-[2][3]$  but it is very difficult to check whether its 6th power is congruent to  $[2]^4[3] + 2[2]^2[3]$  or not. By a help of a machine [1], we can see that there does not exist an  $\alpha \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$  such that

$$(\alpha)^6 \equiv [2]^4[3] + 2[2]^2[3] \pmod{\mathcal{I}_6}$$

even though there do exist a symmetry of order 6 for  $6_1$ .

**4.2. Discussion.** The dual of a three-connected planar cubic graph is a triangulation. In particular, the dual of prime webs are three-connected planar Eulerian triangulations. The number of these triangulations of a fixed vertices can be computed by a C-program "plantri.c" [3] and it has been computed up to 68 vertices [4].

**Conjecture 4.3.** *Let  $k_n$  be the number of prime webs of  $2n$  vertices. Then,*

$$\lim_{n \rightarrow \infty} \frac{k_n}{k_{n-1}} = 3.829....$$

One can ask what are the sufficient information to classify all prime webs.

**Conjecture 4.4.** *The quantum  $\mathfrak{sl}(3)$  invariant, polygonal descriptions and circularness completely determines all primes webs.*

Prime Web	Quantum $\mathfrak{sl}(3)$ invariant	Polygonal descriptions	Circularness
$4_1$	$2[2]^2[3]$	$4 + 4$ $4 + 4$ $4 + 4$	Yes
$6_1$	$[2]^4[3] + 2[2]^2[3]$	$4 + 4 + 4$ $4 + 4 + 4$ $6 + 6$	Yes
$7_1$	$-4[2]^3[3]$	$4 + 4 + 6$ $4 + 4 + 6$ $4 + 4 + 6$	Yes
$8_1$	$[2]^6[3] + [2]^4[3] + 2[2]^2[3]$	$4 + 4 + 4 + 4$ $4 + 4 + 4 + 4$ $8 + 8$	Yes
$8_2$	$3[2]^4[3] + 2[2]^2[3]$	$4 + 4 + 4 + 4$ $4 + 6 + 6$ $4 + 6 + 6$	Yes
$9_1$	$-[2]^5[3] - 6[2]^3[3]$	$4 + 4 + 4 + 6$ $4 + 4 + 4 + 6$ $6 + 6 + 6$	Yes
$9_2$	$-2[2]^5[3] - 4[2]^3[3]$	$4 + 4 + 4 + 6$ $4 + 4 + 4 + 6$ $4 + 6 + 8$	Yes
$10_1$	$[2]^8[3] + [2]^6 + [2]^4[3] + 2[2]^2[3]$	$4 + 4 + 4 + 4 + 4$ $4 + 4 + 4 + 4 + 4$ $10 + 10$	Yes
$10_2$	$8[2]^4[3][3]$	$4 + 4 + 4 + 8$ $4 + 4 + 4 + 8$ $4 + 4 + 6 + 6$	Yes

TABLE 1. Quantum  $\mathfrak{sl}(3)$  invariant, polygonal descriptions and circularness of prime webs up to 20 vertices.

For  $n \geq 4$ , there is the quantum  $\mathfrak{sl}(n, \mathbb{C})$  invariants of webs, which are cubic weighted directed planar graphs [12]. It would be very interesting how one can overcome the technical difficulty arose on the weights of edges.

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Prime Web	Quantum $\mathfrak{sl}(3)$ invariant	Polygonal descriptions	Circularness
$10_3$	$[2]^6[3] + 5[2]^4[3] + 2[2]^2[3]$	$4 + 4 + 4 + 4 + 4$ $4 + 4 + 6 + 6$ $6 + 6 + 8$	Yes
$10_4$	$8[2]^4[3]$	$4 + 4 + 4 + 8$ $4 + 4 + 6 + 6$ $4 + 4 + 6 + 6$	Yes
$10_5$	$6[2]^4[3][3] + 3[2]^4[3] + 2[2]^2[3]$	$4 + 4 + 4 + 4 + 4$ $4 + 4 + 6 + 6$ $4 + 8 + 8$	Yes
$10_6$	$7[2]^4[3] + 2[2]^2[3]$	$4 + 4 + 6 + 6$ $4 + 4 + 6 + 6$ $4 + 4 + 6 + 6$	No
$10_7$	$[2]^6[3] + 5[2]^4[3] + 2[2]^2[3]$	$4 + 4 + 6 + 6$ $4 + 4 + 6 + 6$ $4 + 4 + 6 + 6$	No
$10_8$	$8[2]^4[3]$	$4 + 4 + 6 + 6$ $4 + 4 + 6 + 6$ $4 + 4 + 6 + 6$	No

TABLE 2. Table I continued.

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DEPARTMENT OF MATHEMATICS, KYUNGPOOK NATIONAL UNIVERSITY, TAEGU 702-201 KOREA  
*E-mail address:* dongseok@mail.knu.ac.kr

DEPARTMENT OF MATHEMATICS, YEUNGNAM UNIVERSITY, KYONGSAN, 712-749, KOREA  
*E-mail address:* julee@yu.ac.kr